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Birational equivalence of reduced graphs

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Abstract

We study a canonical desingularization process for oriented reduced graphs. We use it to give an arithmetical characterization for these graphs by means of sequences of natural numbers, based on the representation of a partial ordering by its maximal chains. For such graphs we define, in analogy with Algebraic Geometry, similar tools and language as in birational geometry. We study a class of birationally equivalent graphs giving it a graph structure and describing, in an explicit way, a canonical graph in the class with a minimal number of points.
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0. Introduction

In this paper we are interested in the class of reduced graphs, representing antisymmetric transitive digraphs with no loops nor multiple arcs. It is well known [1] that every digraph (X, G) of this class also represents a partial ordering $(X, \leq)$ where $x \leq y$ if and only if $(x, y) \in G$. Both structures can also be seen as a T_0-topological space over X (x is adherent to $\{y\}$ if and only if $x \leq y$) for which the family $\{\{\bar{x}\} \mid x \in X\}$ is a subbase of closed sets. The morphisms between partial orderings are the continuous maps between the correspondent topological spaces.

Our initial motivation has been to study the graphs associated with geometrical configurations, i.e. finite sets of irreducible subvarieties of a given variety, the graph structure corresponding to the inclusion ordering. Looking at the methods of resolution of singularities [3, 6, 7] we introduce in [5] a notion of blowing up valid for oriented acyclic graphs, where in the case of graphs of good geometrical configurations, the transformed graph is associated with the blown up configuration. Moreover, as in

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geometry, we also proved in [5] that there exists a desingularization process for oriented acyclic graphs which allows us to better understand their combinatorial structure.

When the graph is reduced the process of desingularization is simpler (see Section 1 below) and it presents special features. The purpose of this paper is to use this process to classify reduced graphs by means of a natural relation which, for obvious reasons, we call birational equivalence.

In Section 1 we describe the process of desingularization in a forest (Z,H) for reduced graphs using the fact that a partial ordering is uniquely determined by its maximal chains [4]. We see how ordered labelings on a graph (X,G) induce ordered labelings on its desingularizations. These labelings show how the points of the desingularized graph (Z,H) are uniquely determined by the paths on the graph (X,G) ending in maximal points. In Section 2 we show how the label set on the forest (Z,H) determines the original graph and we use this fact to characterize arithmetically the possible ordered label sets on a forest in such a way that we can characterize arithmetically a reduced graph in a natural way. In Section 3 we count the number of ways of doing it.

In Section 4 we describe the opposite process to desingularization and we give a characterization of the conditions under which it is possible to recover a reduced graph from its desingularization, the forest (Z,H). Reduced graphs with isomorphic desingularizations are called birationality equivalent, a birational morphism being a graph morphism inducing an isomorphism between their desingularizing forests. The terminology is, here, completely analogous to that in algebraic geometry and the theory is similar to that of algebraic curves [2]. Reduced graphs in a class module birational equivalence can be partially ordered, the arcs being the birational morphisms (Section 5). If (Z,H) is the common forest for the class, our general objective is to study this ordered set $\mathrm{Iso}(Z,H)$ that contains one maximal element, the forest (Z,H) itself, but in general it has several minimals. Section 6 is devoted to proving the existence of a canonical contraction with a minimal number of points and arcs representing the class $\mathrm{Iso}(Z,H)$. This graph is explicitly described and it is, in a certain sense, the smallest object containing all information in the forest. In Section 7 we give some additional properties of the structure on $\mathrm{Iso}(Z,H)$ to clarify it.

1. Desingularization of reduced graphs. Basic concepts and notations

Let us introduce the basic concepts and notations which we will use in the paper.

By a *graph* we mean a couple (X,G) where X is a finite set and $G \subset X_x X - \{(x,x) \mid x \in X\}$. The elements in X and G are called *points* and *arcs*, respectively, and if $(x,y) \in G$ then x is said to be *adjacent* to y.

A *labeling* of X by the label set E is a bijective map $\underline{x} : E \to X$, $\underline{x}(e)$ being denoted by x_e for any $e \in E$. For a *labeled graph* we mean a graph (X,G) with a labeling on the set X.

For two given graphs (X,G), (X',G') a *graph morphism* is a mapping $\Delta : X \to X'$ such that for every arc $(x,y) \in G$, one has either $\Delta(x) = \Delta(y)$ or $(\Delta(x), \Delta(y)) \in G'$. The morphism is an *isomorphism* if it has an inverse, i.e. if Δ is bijective and $(x,y) \in G$ iff $(\Delta(x), \Delta(y)) \in G'$. By a *non-labeled graph* we will mean an equivalence class of isomorphic graphs.

A *sequence* of points in X will be considered as a mapping $s : \{1,2,\ldots,q\} \to X$, $q \geq 2$, or alternatively as $s(1)\ s(2)\ \cdots\ s(q)$ or $x_1 x_2 \cdots x_q$. A sequence such that $(s(i), s(i+1)) \in G$ is called a *walk* joining x_1 with x_q in (X,G) of *length* $q-1$. When $x_1 = x_q$ the walk is said to be *closed*, if s is injective the walk is called a *path*, and if s is closed and injective on $\{1,2,\ldots,q-1\}$ the walk is called a *cycle*. A graph is said to be *acyclic* if it has no cycles. Finally, a *semiwalk* in (X,G) we mean a sequence such that for any i, $1 \leq i \leq q-1$, one has either $(s(i),(i+1)) \in G$ or $(s(i+1), s(i)) \in G$.

By a *subgraph* of (X,G) we will mean a graph (Y,H) with $Y \subset X$ and $H \subset G$. The subgraph will be called a *partial graph* when $Y = X$. On the other hand, by the *induced subgraph* we will mean the graph $(Y, G/Y)$ where $G/Y = G \cap (Y \times Y)$.

For a graph (X,G) the *dual graph* is (X, G^{d}) where $G^{\mathrm{d}} = \{(x,y) \mid (y,x) \in G\}$.

For a point $x \in X$ we will consider the sets $\bar{x}$ (resp. x^*) consisting of x and those points $y \in X$ such that there exists a path joining y to x (resp. x to y). The set $\bar{x}$ will be called the *closure* of x.

Acyclic graphs have some interesting properties.

Firstly, for an acyclic graph there exists at least one point x (resp. y) such that $\bar{x} = \{x\}$ (resp. $y^* = \{y\}$). Such a point will be called a *minimal* (resp. *maximal*) in the graph. Moreover, the points in an acyclic graph (X,G) can be distributed by *levels* N_0, $N_1, \ldots$ as follows:

$$N_0 = \{x \in X \mid x \text{ is minimal in } (X,G)\}$$

and recursively for $p \geq 1$

$$N_p = \left\{ x \in X - \bigcup_{i=0}^{p-1} N_i \mid x \text{ is minimal in } \left(X - \bigcup_{i=0}^{p-1} N_i, G/X - \bigcup_{i=0}^{p-1} N_i \right) \right\}.$$

Thus one has a partition of X, $X = \bigcup_{p=0}^{k} N_p, k$ being the *dimension* of the graph, i.e. the last index such that $N_k \neq \Phi$. The equivalence relationship associated to this partition is the "equality of heights between points of X" where the *height* of $x \in X$ is $h(x) =$ the length of the largest path ending in x. So $x \in N_p$ if and only if $h(x) = p$.

Secondly, we have the following characterization of acyclic graphs [5]: a graph (X,G), with $\operatorname{card}(X) = n$, is acyclic if and only if, there exists a labeling of X by the label set $E = \{1,\ldots,n\}$ such that if $(x_i, x_j) \in G$ then $i < j$. To prove it, it is enough to consider the above partition and to give a bijection $\underline{x} : E \to X$ such that, for every $m \in E$,

$$\underline{x}(m) = x_m \in N_p \Leftrightarrow \sum_{i=0}^{p-1} n_i < m \leq \sum_{i=0}^{p} n_i,$$

where $n_i = \operatorname{card}(N_i)$ and $0 < m \leq n_0$, if $p = 0$.

On the other hand, we can consider the set of maximal points M in the acyclic graph (X,G). A path $s:\{1,\dots,q\}\to X$ will be said to be a *path with maximal end* if $s(q)\in M$, and furthermore if $s(1)\in N_0$ then the path will be called *maximal*. The morphisms which take maximal points in maximal points will be called *dominant*.

For a graph (X,G) the equivalence relationship "$x\equiv y$" iff x and y are in a "semi-walk" gives rise to the partition of X into connected components $X=X_1\cup\cdots\cup X_r$. Properly speaking the *connected components* are the induced subgraph $(X_i,G/X_i)$.

For a non-maximal point $x\in X-M$ in a graph (X,G) the *outdegree* is defined as $\mathrm{Od}(x)=\mathrm{Card}\{y\mid (x,y)\in G\}$. The point x will be said to be *regular* if $\mathrm{Od}(x)=1$ and *singular* otherwise. A *tree* is an acyclic graph for which there exists a maximal point $y\in X$ such that for every $x\neq y$ there is a unique path $x_1\cdots x_q$ with $x_1=x$ and $x_q=y$. The point y is unique and a *tree* has no singular points. The graphs with no singular points are exactly the *forests*, i.e. the graphs whose connected components are trees.

A graph (X,G) is said to be *transitive* if for every pair of arcs (x,y), (y,z) with $x\neq z$, then (x,z) is an arc. The graph is *antisymmetric* if it has no pair of symmetric arcs. Antisymmetry and acyclicity are equivalent properties for transitive graphs (if (x,y) and (y,x) are arcs, then xyx is a cycle; and conversely, if $x_1\cdots x_q$ is a cycle, then both (x_1,x_{q-1}) and (x_{q-1},x_1) are arcs).

Often a partial ordering (and so its homologous antisymmetric transitive graph and T_0-topological space) is represented by a picture in which those arcs which are redundant by transitivity are dropped. The homologous graph of this picture will be called "reduced" and it is also known as Hasse diagram.

A partial graph (X,G^{r}) of an antisymmetric transitive graph (X,G) will be said to be *reduced* if $G^{\mathrm{r}}=G-\{(x,y)\in G\mid$ there exists a path $x_1\cdots x_q$ in (X,G) with $q\geq 3$ such that $x_1=x$ and $x_q=y\}$. Without mention of the original partial ordering, a graph (X,G) is *reduced* if it is acyclic and if (x,y) is an arc, then there is no other path joining x with y.

In this paragraph we shall transform a reduced graph into one without singular points. This transformation will affect each singular point and consists in the removal of the arcs that leave from it. Each arc is preserved, maintaining the induced subgraph in the closure of the singular point on its lower extreme. This transformation is performed in an orderly way, by levels from N_0 to N_k, and can be interpreted as a well arranged desingularization of the reduced graph.

Definition 1.1. A graph (X,G) is said to be an *E-orderly graph* if X is labeled by a totally ordered label set E and the bijection $\underline{x}:E\to X$, with $\underline{x}(i)=x_i$, verifies that if $(x_i,x_j)\in G$ then $i<j$. If $E=\{1,\dots,n\}$ we will say that the E-orderly graph is *naturally ordered.*

Observation 1.2. Note that if (X,G) is an E-orderly reduced graph, the partial ordering associated with (X,G) is a less fine order than the image order of that of E through the bijection $\underline{x}$ and, therefore, to give an E-order on a reduced graph (X,G) is essentially to give a total order on X finer than the partial ordering associated with the graph (X,G).

Definition 1.3. The *desingularized* of a reduced graph (X, G) is the graph $(\tilde{X}, \tilde{G})$ where $\tilde{X}$ is the set of paths with maximal end together with the set of maximal points M and $\tilde{G} = \{(x, y) \in \tilde{X} \times \tilde{X} \mid x \notin M$ and y is the path obtained from x by deleting the first element$\}$.

If the reduced graph (X, G) is labeled by means of $E = \{1, \dots, n\}$ by the bijection $\underline{x} : E \to X$, then the desingularized graph $(\tilde{X}, \tilde{G})$ will be considered to be labeled by $\tilde{E} = \{A \subset E \mid \underline{x}_A \in \tilde{X}\}$ by means of the bijection $\tilde{x} : \tilde{E} \to \tilde{X}$ given by $\tilde{x}(A) = x_{i_1} \cdots x_{i_q} \in \tilde{X}$, where $\underline{x}_A$ denotes the restriction of $\underline{x}$ to the naturally ordered set $A = \{i_1 < \cdots < i_q\}$. Moreover, if the reduced graph (X, G) is naturally ordered by $E = \{1, \dots, n\}$, then $\tilde{E}$ is totally ordered by the lexicographic ordering and therefore, the desingularized graph $(\tilde{X}, \tilde{G})$ is an $\tilde{E}$-orderly graph.

Proposition 1.4. *The desingularized graph $(\tilde{X}, \tilde{G})$ of a reduced graph is a forest with m trees, where $m = \mathrm{Card}(M)$.*

Proof. If $(x, y) \in \tilde{G}$ and $x = x_{i_1} \cdots x_{i_q}$ then $y = x_{i_2} \cdots x_{i_q}$, so $\mathrm{Od}(x) = 1$ and there is no singular point in $(\tilde{X}, \tilde{G})$. It is also clear that in $(\tilde{X}, \tilde{G})$ there are many maximal points as in (X, G). □

The map $\pi : (\tilde{X}, \tilde{G}) \to (X, G)$ given by $\pi(x_1 \cdots x_q) = x_1$ is a dominant morphism and will be called *desingularization* of the reduced graph (X, G).

2. Arithmetical characterization of reduced graphs

Each non-maximal point of the reduced graph (X, G) gives rise to a new point in the desingularized $(\tilde{X}, \tilde{G})$ for each one of the paths with maximal end in (X, G) starting from it.

Proposition 2.1. *The labels of the level $\tilde{N}_0$ in $(\tilde{X}, \tilde{G})$ describe the graph from the start (X, G).*

Proof. It is clear that $x_i \in X$ if and only if x_i is in the label of some minimal point of $(\tilde{X}, \tilde{G})$ and $(x_i, x_j) \in G$ if and only if x_i and x_j are consecutive (in that order) in the label of some point of $\tilde{N}_0$ in $(\tilde{X}, \tilde{G})$. Thus,

$$X = \bigcup_{x \in \tilde{N}_0} \{x_i \text{ being in the label of } x \in \tilde{N}_0 \subset \tilde{X}\},$$

$$G = \{(x_i, x_j) \mid \exists x = x_{j_1} \cdots x_{j_r} x_{j_{r+1}} \cdots x_{j_q} \in \tilde{N}_0 \text{ in } (\tilde{X}, \tilde{G}) \text{ with } i = j_r \text{ and } j = j_{r+1}\}.$$

□

We can ask under what conditions a collection of labels defines a reduced graph (X, G), i.e. "when $\mathbf{A} \subset \mathbf{P}(E)$ represents a reduced graph (X, G), $\mathbf{A}$ being the label set of the minimal points of its desingularized $(\tilde{X}, \tilde{G})$?" The labels $A \in \mathbf{P}(E)$ should be interpreted as maximal paths in the graph (X, G).

Theorem 2.2 (Determination of a reduced graph by means of labels). *Let* $E = \{1,\ldots,n\}$ *and* $\mathbf{A} \subset \mathbf{P}(E)$. *For every* $A \in \mathbf{A}$ *we assume that* A *is naturally ordered.* $\mathbf{A}$ *determines a reduced graph* (X,G), $\mathbf{A}$ *being the label set of the minimal points of its desingularized* $(\tilde{X},\tilde{G})$, *if and only if the following conditions hold:*

(a) *E is the joint of the elements* $A \in \mathbf{A}$.

(b) *If p is the first element of A and $p \in B$, with $B \in \mathbf{A}$, then p is the first element of B; and if p is the last element of A and $p \in B$, with $B \in \mathbf{A}$, then p is the last element of B.*

(c) *If $p,q \in A$ such that q does not follow p in A and $p,q \in B$, with $B \in \mathbf{A}$, then q does not follow p in B.*

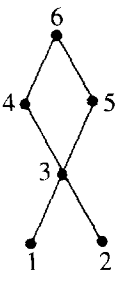
In this case $X = E$ and $(p,q) \in G$ if and only if there exists $A \in \mathbf{A}$, with $p,q \in A$, such that q follows p in A.

Proof. The conditions (a)–(c) are clearly necessary. Conversely, the natural ordering allows us to interpret A as a path (injective sequence); (b) means that these paths are maximals (they start in the level N_0 and end in a maximal point) and (c) imposes the reduction of the graph (the graph is acyclic because A is naturally ordered).

Therefore we can interpret $\mathbf{A}$ as a collection of maximal paths of a reduced graph (X,G). □

Remark 2.3. The set $\mathbf{A}$ only contains a collection of maximal paths sufficient "to cover" G, but $\mathbf{A}$ does not represent all maximal paths of (X,G). For example, $\mathbf{A} = \{\{1,3,4,6\},\{2,3,5,6\}\}$ covers the reduced graph

$(X,G) =$



however the maximal path set is $\mathbf{A} \cup \{\{1,3,5,6\},\{2,3,4,6\}\}$.

As a consequence of the above theorem, the following condition guarantees that the labels of $\mathbf{A}$ represent all maximal paths of a reduced graph (X,G):

(d) *Let* $N_0 = \{$*first elements of the sets* $A \in \mathbf{A}\}$ *and* $M = \{$*last elements of the sets* $A \in \mathbf{A}\}$. *If* $B = \{p_1,\ldots,p_m\}$ *is a naturally ordered subset of* E *such that* $p_1 \in N_0$, $p_m \in M$ *and for every* $j = 1,\ldots,m-1$ *there exists* $A_j \in \mathbf{A}$ *with* p_j, $p_{j+1} \in A_j$ *and such that* p_{j+1} *follows* p_j *in* A_j, *then* $B \in \mathbf{A}$.

In this way we have the following result.

Theorem 2.4 (Characterization of label sets of reduced graphs). *Let* $E=\{1,\dots,n\}$ *and* **A** *be a subset of parts (naturally ordered) of E verifying the conditions* (a)–(d). *Then, there exists a unique reduced graph* (X,G) *with* $\operatorname{card}(X)=n$ *and a unique labeling* $\underline{x}:E\to X$ *of* X *by the label set E such that* (X,G) *is E-ordered and* **A** *is the label set of the level* $\tilde{N}_0$ *of its desingularized* $(\tilde{X},\tilde{G})$.

Proof. It will be enough to prove that every path starting in the level N_0 and ending in a maximal point of (X,G) is an element of **A**. Let B be the naturally ordered set corresponding to such a path; clearly B satisfies the condition (d) and therefore $B\in\mathbf{A}$. □

In Section 1 we have characterized an acyclic graph (X,G) by means of the existence of a labeling of X by $E=\{1,\dots,n\}$ under the condition $(x_i,x_j)\in G\Rightarrow i<j$. If, moreover, this labeling verifies $x_m\in N_p\Leftrightarrow \sum_{i=0}^{p-1}n_i<m\le\sum_{i=0}^{p}n_i$ then the graph (X,G) is naturally ordered. But the reciprocal is not generally true.

Definition 2.5. Let E be a totally ordered finite set, (X,G) an E-ordered acyclic graph and $X=\bigcup_{p=0}^{k}N_p$ its partition in levels. The graph (X,G) is said to be *E-ordered by levels* if $x_i\in N_p$, $x_j\in N_q$, with $p<q$, then $i<j$. In the particular case $E=\{1,\dots,n\}$ the graph (X,G) is said to be *naturally ordered by levels*. Thus, with notations as above, one has (X,G) is naturally ordered by levels if and only if $\sum_{i=0}^{p-1}n_i<m\le\sum_{i=0}^{p}n_i$, for any $x_i\in N_p$, $i\in E$, and for any $p=0,1,\dots,k$ (it is understood that $\sum_{i=0}^{p-1}n_i=0$ when $p=0$).

Definition 2.6. Assume that E is $\{1,\dots,n\}$ and that the set $\mathbf{A}\subset\mathbf{P}(E)$ satisfies the above mentioned conditions (a)–(d). Then, for every $i\in E$, the maximum number of places occupied by i in the sets $A\in\mathbf{A}$ which contain it will be called *length* of i and will be denoted by $l(i)$. It is clear that the set **A** determines a naturally ordered graph verifying $x_i\in N_p\Leftrightarrow l(i)=p$.

Theorem 2.7 (Characterization of labels ordered by lengths). *Let* $E=\{1,\dots,n\}$ *and* $\mathbf{A}\subset\mathbf{P}(E)$ *with every* $A\in\mathbf{A}$ *naturally ordered.* **A** *is the collection of labels of the minimal points of the desingularized of a reduced graph* (X,G) *naturally ordered by levels if and only if* **A** *verifies the conditions* (a)–(d) *and*

(e) $l(i)<l(j)\Rightarrow i<j$.

Proof. After Theorem 2.4 it is enough to bear in mind that the equality of heights between points of X is the equivalence relationship associated to the partition of X in levels. Thus $x_i\in N_p\Leftrightarrow h(x_i)=p$. □

3. Number of naturally ordered labelings for a non-labeled reduced graph

The preceding theorems allow us to recover a non-labeled reduced graph from a collection of labels in its desingularized graph; however, the same non-labeled reduced graph will generally have several of these label sets assigned, since there exists more than one natural ordering on a graph and, furthermore, more than one natural ordering by levels.

We can now study the number of possible natural orderings, or natural ordering by levels, or collections of labels according to the above theorems.

Let $E=\{1,\dots,n\}$ and $\mathbf{A}\subset\mathbf{P}(E)$ under the conditions (a)–(d) and let (X,G) be the associated non-labeled reduced graph. We will denote the permutation group of E by S_n and by B the subgroup of S_n formed by the permutations $b : E \to E$ such that, for every $A \in \mathbf{A}$, if $A=\{i_1<\cdots<i_q\}$ then $b(i_1)<\cdots<b(i_q)$. Let $B_0\subset B$ be the subgroup of B formed by those permutations $b\in B$ such that $b(A)\in\mathbf{A}$, for any $A\in\mathbf{A}$.

Proposition 3.1. (1) *The number of natural orderings on the non-labeled reduced graph* (X,G) *is equal to the cardinal of* B.

(2) *The automorphism group of the non-labeled reduced graph* (X,G) *is isomorphic to the group* B_0. *In particular*, $\mathrm{Card}\,(\mathrm{Aut}\,(X,G))=\mathrm{Card}\,(B_0)$.

(3) *The number of collections* $\mathbf{A}\subset\mathbf{P}(E)$ *under the conditions* (a)–(d) *determining the non-labeled reduced graph* (X,G) *is equal to* $\mathrm{Card}\,(B)/\mathrm{Card}\,(B_0)$.

Proof. In fact, it is clear that if $\underline{x} : E\to X$ is a labeling providing a natural ordering on (X,G) and if $b\in B$, then $\underline{x}_0 b : E\to X$ is, again, a labeling providing a natural ordering on (X,G). Reciprocally, if $\underline{x} : E\to X$, $\underline{x}' : E\to X$ are two labelings providing natural orderings on (X,G), then the bijection $b : E\to E$ given by $b=\underline{x}_0^{-1}\underline{x}'$ is an element of B. Therefore, with a fixed collection $\mathbf{A}$ under the conditions (a)–(d) and the associated labeling $\underline{x} : E\to X$ (providing a natural ordering on (X,G)), then the correspondence that associates the element $\underline{x}_0^{-1}\underline{x}'$ of B to each natural ordering $\underline{x}' : E\to X$, is a bijection. This proves (1).

(2) is derived from the fact that if $\underline{x} : E\to X$ is a labeling of (X,G) corresponding to the collection $\mathbf{A}$, then each automorphism $\sigma : (X,G)\to(X,G)$ determines a bijection $b : E\to E$ with the property $b\in B$ and $b(A)\in\mathbf{A}$, for any $A\in\mathbf{A}$. Conversely, if $b\in B$ is a permutation with the property $b(A)\in\mathbf{A}$, for any $A\in\mathbf{A}$, then b determines a bijection $\sigma : X\to X$, which is, clearly, an isomorphism when we consider $\sigma : (X,G)\to(X,G)$ as a graph application.

To prove (3) it is enough to bear in mind that if $\mathbf{C}$ is $\{\mathbf{A}\subset\mathbf{P}(E)\}$ verifying the conditions (a)–(d) and determining the same reduced graph (X,G), then the group B acts transitively on $\mathbf{C}$ and, therefore, this action has only one orbit $\mathbf{C}$. The isotropy group of the element $A\in\mathbf{C}$ is B_0, and therefore $\mathrm{Card}\,(\mathbf{C})\mathrm{Card}\,(B_0)=\mathrm{Card}\,(B)$. □

If we consider natural orderings by levels we can prove a similar proposition.

Now, let $\mathbf{A}\subset\mathbf{P}(E)$ be under the conditions (a)–(e) and let (X,G) be the non-labeled reduced graph associated to $\mathbf{A}$. Let $\mathbf{C}$ be the subgroup of S_n formed by those permutations $b\in S_n$ such that for every $A\in\mathbf{A}$, if $A=\{i_1<\cdots<i_q\}$ then $b(i_1)<\cdots<b(i_q)$ and furthermore for every $i\in E$, $l(i)=l(b(i))$. And let $\mathbf{C}_0$ be the subgroup of $\mathbf{C}$ formed by those $b\in\mathbf{C}$ such that $b(A)\in\mathbf{A}$, for any $A\in\mathbf{A}$. Then, we have

Proposition 3.2. (1) *The number of natural orderings by levels on the non-labeled reduced graph* (X,G) *is equal to the* $\mathrm{Card}(\mathbf{C})$.

(2) *The automorphism groups of the non-labeled graph* (X,G) *is isomorphic to the group* $\mathbf{C}_0$. *In particular,* $\mathrm{Card}(\mathrm{Aut}(X,G)) = \mathrm{Card}(\mathbf{C}_0)$.

(3) *The number of sets* $\mathbf{A} \subset \mathbf{P}(E)$ *subject to the conditions* (a)–(e) *determining the same non-labeled reduced graph* (X,G) *is equal to* $\mathrm{Card}(\mathbf{C})/\mathrm{Card}(\mathbf{C}_0)$.

The proof is identical to that of the previous proposition.

Remark 3.3. If $X = \bigcup_{p=0}^{k} N_p$ is the decomposition of the non-labeled reduced graph (X,G) in levels and n_p is the cardinal of N_p, then $\mathbf{C}$ is isomorph to the group $S_{n_0} \times \cdots \times S_{n_k}$ and, in particular, $\mathrm{Card}(\mathbf{C}) = n_0! \cdots n_k!$.

4. Contraction of a forest to a reduced graph

We shall first show how a labeling in (X,G) induces a "prelabeling" in its desingularized forest $(\tilde{X},\tilde{G})$ which can be done by using the same label set and in such a way as to enable the recuperation of the original structure of (X,G).

Definition 4.1. A *prelabeling* on a forest (Z,H) by the prelabel set E is a surjective map $p: Z \to E$. For every $x \in Z$, $p(x)$ is the *prelabel* of x.

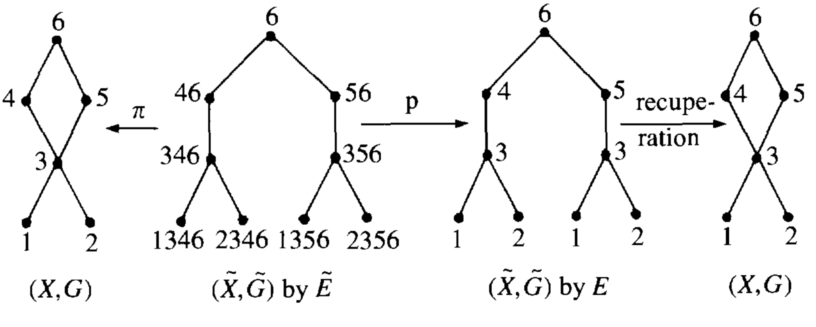
Proposition 4.2. *Let* (X,G) *be a reduced graph labeled by* $E = \{1,\dots,n\}$, *with* $\underline{x}(i) = x_i$, *and let* $(\tilde{X},\tilde{G})$ *be its desingularized forest labeled by* $\tilde{E}$. *The following properties hold:*

(*a*) *The map* $p:\tilde{X} \to E$ *such that* $p(x_{i_1}\dots x_{i_q}) = i_1$ *for any* $x_{i_1}\dots x_{i_q} \in \tilde{X}$, *is a prelabeling on* $(\tilde{X},\tilde{G})$ *by* E.

(*b*) *If* K *is the arc set given by*

$$K = \{(i,j) \in E_x E \mid \textit{there exists } (x,y) \in \tilde{G} \textit{ with } p(x) = i \textit{ and } p(y) = j\}$$

then the labeling bijection $x: E \to X$ *is a graph isomorphism between* (E,K) *and* (X,G).



We shall now demostrate a natural way of finding reduced graphs with a given desingularized forest (up to isomorphism). With a non-labeled forest fixed, the idea is to give (not necessarily different) labels to its points such that the label identification allows us to find the required reduced graph as a quotient. The next definition determines exactly our objective and imposes conditions on the prelabeling.

Definition 4.3. Let (Z,H) be a forest prelabeled by E by map $p:Z \to E$. and $K=\{(i,j)\in E_xE \mid \text{there exists } (x,y)\in H \text{ with } p(x)=i \text{ and } p(y)=j\}$. The surjection p is said to be a *contractive map* on the forest (Z,H) if (E,K) is a reduced graph labeled by E such that its desingularized $(\tilde{E},\tilde{K})$ prelabeled by E is the forest (Z,H), i.e. $(\tilde{E},\tilde{K})$ and (Z,H) are isomorphic forests by means of an isomorphism preserving the prelabelings by E in both forests. In this case, (E,K) will be called the *p-contraction* of (Z,H).

Notice that Proposition 4.2 assures that every reduced graph labeled by E is the p-contraction of its desingularized forest for the contractive map $p(x_{i_1}\cdots x_{i_q})=i_1$ and so a forest admits as many p-contractions as reduced graphs are desingularized by it.

Now we try to give the conditions in which that a forest prelabeling leads to a p-contraction. The key is to know when two different points in the forest can have or not the same prelabel. The following observations about the relationship between a reduced graph and its desingularized forest give some necessary conditions for this prelabeling. First let us give a useful definition.

Definition 4.4. Let (Z,H) be a forest and M its maximal set. For every pair $x,y\in Z-M$, x is said to *cover* y if there exists a path joining y with that which follows x. This relation is a preordering and two points have the same followers if one covers the other one, and conversely.

Observations 4.5. (1) The desingularized forest has as many trees as maximal points in the reduced graph, so the maximals must carry different prelabels.

(2) If a forest point x covers another point y, then x and y must have different prelabels.

(3) Since the "desingularization" of a singular point x yields several copies of $(\bar{x},G/\bar{x})$, not connected among themselves in the forest $(\tilde{X},\tilde{G})$, we must classify the non-connected isomorphic induced subgraphs in the forest. Given two of them $(\bar{x},G/\bar{x})$ and $(\bar{y},G/\bar{y})$, the points of the sets $\bar{x}-\{x\}$ and $\bar{y}-\{y\}$ can have the same prelabeling (same prelabeling set preserved by the isomorphism). In this case the points x and y can carry the same prelabel only if they are not maximals and if each one does not cover the other.

Theorem 4.6. *If (Z,H) is an E-prelabeled forest by map $p:Z\to E$ and $K=\{(i,j)\in E_xE \mid$ there exists $(x,y)\in H$ with $p(x)=i$ and $p(y)=j\}$, then (E,K) is the p-contraction of (Z,H) if and only if, for every $x,y\in Z$ with $p(x)=p(y)$ the following* *conditions hold*:

(1) *x and y are not maximals.*

(2) *Neither x covers y nor y covers x.*

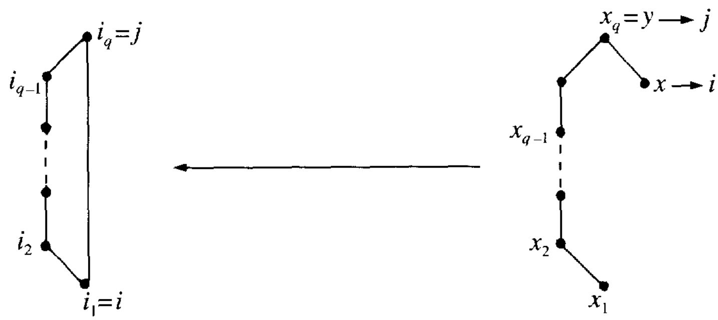
(3) *There exists a graph isomorphism $\Delta:(\bar{x},H/\bar{x}) \to (\bar{y},H/\bar{y})$ such that $p(z)=p(\Delta(z))$, for any $z \in \bar{x}$.*

Proof. We have already commented the necessity of these conditions. Conversely:

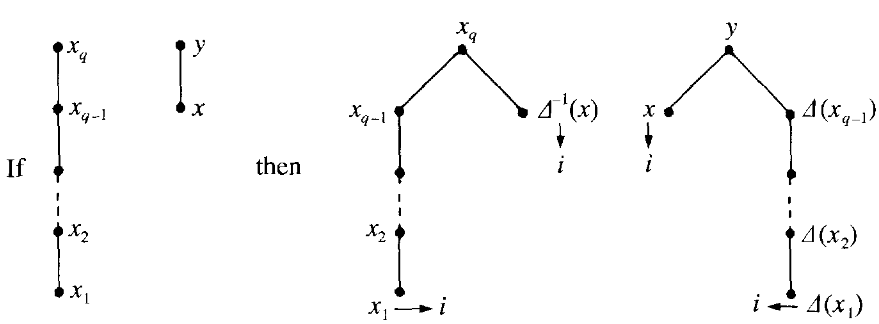
(a) (E,K) is a reduced graph.

If otherwise, there is an arc $(i,j) \in K$ coming from an arc $(x,y) \in H$ with $p(x)=i$, $p(x)=j$, and there is also a path $i_1 i_2 \cdots i_q$ in (E,K) with $i_1=i$, $i_q=j$ and $q>2$. In this path, the arc (i_{q-1},i_q) comes from $(x_{q-1},x_q) \in H$ with $p(x_{q-1})=i_{q-1}$ and $p(x_q)=i_q$, and the arc (i_{q-2},i_{q-1}) comes from $(y_{q-2},y_{q-1}) \in H$ with $p(y_{q-2})=i_{q-2}$ and $p(y_{q-1})=i_{q-1}$, so $\bar{x}_{q-1} \cong \bar{y}_{q-1}$ and therefore the point $x_{q-2}=\Delta^{-1}(y_{q-2})$ is such that $p(x_{q-2})=i_{q-2}$; then there is a path $x_{q-2}x_{q-1}x_q$ in (Z,H) prelabeled by the path $i_{q-2}i_{q-1}i_q$ of (E,K). Recursively, one gets a path $x_1 x_2 \cdots x_q$ in (Z,H) prelabeled by the path $i_1 i_2 \cdots i_q$ with $p(x_k)=i_k$, for any $k=1,\ldots,q$.

If $x_q=y$, then x covers x_1 which contradicts $p(x)=p(x_1)=i$.



If $x_q \neq y$, since they have the same prelabel j, then $\bar{x}_q \cong \bar{y}$ for some Δ. As $x_1 \in \bar{x}_q$, its homologous $\Delta(x_1)$ is in $\bar{y}$ and $p(x_1)=p(\Delta(x_1))=i$, therefore in $\bar{y}$ there are two points, x and $\Delta(x_1)$, with the same prelabel i, such that x covers $\Delta(x_1)$, in contradiction with condition (2).



(b) $(\tilde{E},\tilde{K})=(Z,H)$.

The dominant morphism $p:(Z,H)\to(E,K)$ induces a morphism $\tilde{p}$ between the desingularized forests $(\tilde{Z},\tilde{H})=(Z,H)$ and $(\tilde{E},\tilde{K})$. The morphism $\tilde{p}$ is surjective, because each path in (E,K) induces at least a path in (Z,H). The morphism $\tilde{p}$ is also injective, since if $x_1\cdots x_q$ and $x'_1\cdots x'_q$ are two paths with maximal end, and the same length carrying the same prelabel sequence $i_1\cdots i_q$, then $x_j=x'_j$ for any $j=1,\dots,q$. In fact, x_q and x'_q are maximals with the same label, so $x_q=x'_q$; x_{q-1} and x'_{q-1} have the same following and equal prelabeling, therefore $x_{q-1}=x'_{q-1}$ and so on. Finally, $\tilde{p}$ is a graph morphism by construction, since the correspondence between arcs associated to $\tilde{p}$ is clearly a bijection. □

5. The ordered set of the p-contractions of a forest

Theorem 5.1. *If $\Delta:(X,G)\to(X',G')$ is a dominant morphism of reduced graphs and if $\pi:(\tilde{X},\tilde{G})\to(X,G)$ and $\pi':(\tilde{X}',\tilde{G}')\to(X',G')$ are the respective desingularizations of (X,G) and (X',G'), then there exists a unique dominant morphism $\tilde{\Delta}:(\tilde{X},\tilde{G})\to(\tilde{X}',\tilde{G}')$ such that $\Delta_0\pi=\pi'_0\tilde{\Delta}$.*

Proof. If x is the path $x_1x_2\cdots x_q\in\tilde{X}$ and x' is the sequence $x'=\Delta(x_1)\Delta(x_2)\cdots\Delta(x_q)$ where the repeated points have been removed, then $\Delta(x_q)$ is maximal since Δ is dominant and so $\tilde{\Delta}(x)=x'$ is the dominant morphism that we are looking for. This morphism is unique because if $\Psi:(\tilde{X},\tilde{G})\to(\tilde{X}',\tilde{G}')$ is another morphism with $\Delta_0\pi=\pi'_0\Psi$ and $x=x_1x_2\cdots x_q\in\tilde{X}$ then one has the following two possibilities. If $q=1$, since $\pi'_0\tilde{\Delta}=\pi'_0\Psi$ and the restriction of π' to M is the identity, then $\tilde{\Delta}=\Psi$. If $q>1$, let $y=x_2\cdots x_q$, so by induction hypothesis $\tilde{\Delta}(y)=\Psi(y)$. Since $(x,y)\in\tilde{G}$ then one has either $(\Psi(x),\Psi(y))\in\tilde{G}'$ or $\Psi(x)=\Psi(y)$ and $\Psi(x)$ is a path $A\Delta(x_2)\cdots\Delta(x_q)$. On the other hand $\pi'_0\Psi(x)=\Delta_0\pi(x)=\Delta(x_1)$ so $\Psi(x)$ is a path starting in $\Delta(x_1)$ and therefore $\Psi(x)=\tilde{\Delta}(x)$. □

Corollary 5.2 (Universal property of desingularization). *Let (X,G) be a reduced graph and $\pi:(\tilde{X},\tilde{G})\to(X,G)$ its desingularization. Then, for every forest (Z,H) and every dominant morphism $\Delta:(Z,H)\to(X,G)$ there exists a unique morphism $\Delta':(Z,H)\to(\tilde{X},\tilde{G})$ such that $\pi_0\Delta'=\Delta$. The morphism Δ' is dominant.*

It is sufficient to bear in mind that $(Z,H)=(\tilde{Z},\tilde{H})$ and that its desingularization is the identity. Then $\Delta'=\tilde{\Delta}$.

Definition 5.3. A morphism between reduced graphs $\Delta:(X,G)\to(X',G')$ will be called a *birational morphism* if it induces an isomorphism $\tilde{\Delta}$ between their desingularized forests. The desingularization of a reduced graph is an example of birational morphism.

Proposition 5.4. *Let $\Delta : (X, G) \to (X', G')$ be a birational morphism between reduced graphs. Then*
(a) *The map Δ is surjective.*
(b) *Δ induces a surjective map between the arc sets $\tilde{\Delta} : G \to G'$.*

Proof. If $x' \in X'$ is not the image of any point of X then every path with maximal end starting in x' cannot be the image of any path with maximal end in (X, G) in contradiction with the bijectiveness of $\tilde{\Delta}$. On the other hand, the map carrying the arc $(x, y) \in G$ into the arc $(\Delta(x), \Delta(y)) \in G'$ (notice that $\Delta(x) \neq \Delta(y)$) is also surjective because if $(x', y') \in G'$ is not the image by $\tilde{\Delta}$ of some arc of G, then every path with maximal end in (X', G') including the arc (x', y') could not be in the image of $\tilde{\Delta}$. □

Corollary 5.5. *If (X, G) and (X', G') are reduced graphs and there are birational morphisms $\Delta : (X, G) \to (X', G')$ and $\Delta' : (X', G') \to (X, G)$, then both Δ and Δ' are graph isomorphisms.*

In fact, $\operatorname{Card}(X) = \operatorname{Card}(X')$, $\operatorname{Card}(G) = \operatorname{Card}(G')$ and Δ and Δ' are both surjective morphisms, therefore they are isomorphisms.

After this corollary it is possible to define the following partial ordering.

Definition 5.6. Let (Z, H) be a non-labeled forest and (X, G), (X', G') two non-labeled reduced graphs with desingularized (Z, H). Then we will write $(X, G) \leq (X', G')$ if there exists a birational morphism $\Delta : (X', G') \to (X, G)$.

Next we will study this partial ordering. Clearly, it has a maximal element (Z, H) itself, since a reduced graph is a forest if and only if, it is isomorphic to its desingularized form. In general, this partial ordering will have several minimals. We shall construct a canonical reduced graph with minimal number of points and minimal number of arcs.

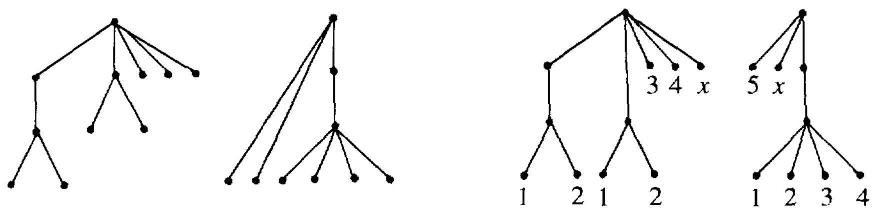
6. Minimal canonical contraction of a forest

6.1. General observations

For every non-labeled forest (Z, H) we will give a prelabel set E and a contractive map p, both canonical. After Theorem 4.6, condition (1) suggests prelabeling the maximal points separately. Condition (3) demands classification of the points by levels, and in each level by their closures. This gives us two indices to each prelabel. Condition (2) is opposite to the others: two points in the forest with the same prelabel must be in the same level and they will have the same closure, but their respective followers are different and they are placed in arbitrary levels. Moreover, since one of these points is not covering the other, they are adjacent to different maximal paths. Thus, in order to describe the conditions for a canonical contraction with minimal Card(E) it is necessary to enter the prelabeling by levels.

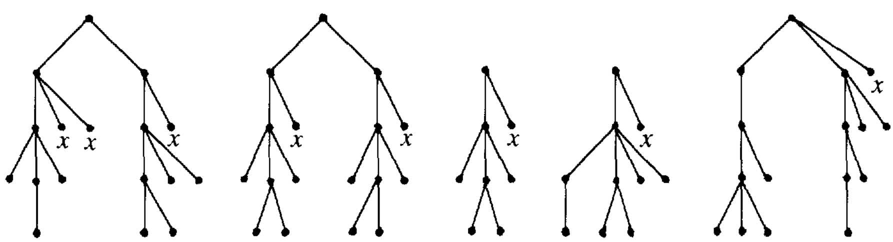
6.2. *First observations about the prelabeling of the level N_0*

In a graphic representation of a forest we will place the points of the same level at the same height in the drawing. All the N_0 points have the same closure so it is only possible to distinguish them by the level of their respective followers. Thus, in this particular case, we will place the N_0 points joined to their followers by an arc of "height" one.



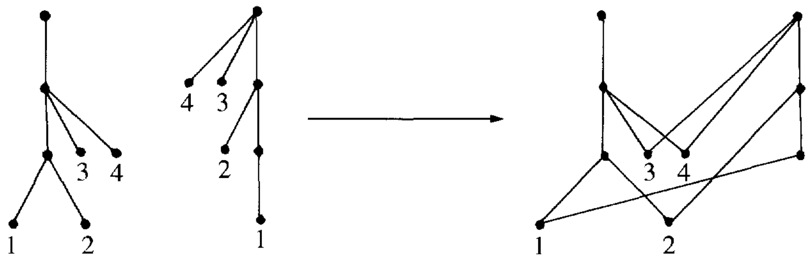
wrong correct

Among all points covered by the point x in the correct forest, only those points joined to the "support path" by exactly one arc must be counted. Thus, for the tree on the left, 5 prelabels are enough while, for the tree on the right, 6 are needed. Then the forest does not need more than 6 different prelabels. In the following picture all prelabeling points with x cover the same number (5) of N_0 points with different prelabels.

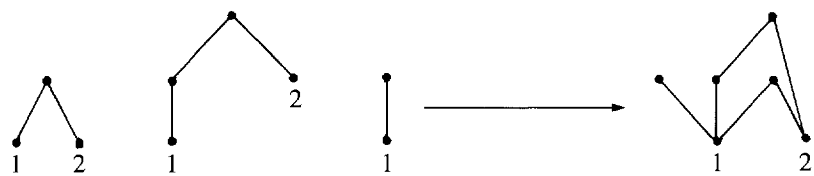


The questions are: How to define this number of prelabels? How to choose the "branch" of the tree with prelabeling "support" for the other "branches"?

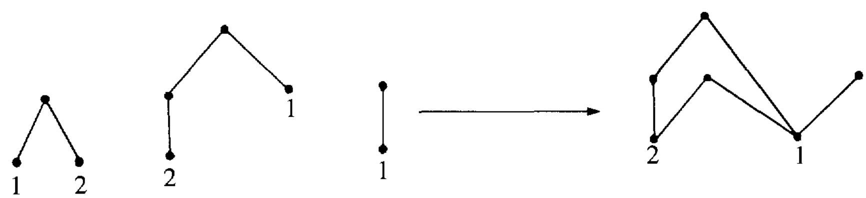
Since we must *identify points with followers in different levels*, we can make it *orderly*; for example, for every "branch" of each tree we can order the necessary prelabels in N_0 *according to the levels of their followers* and then we can make the identification preserving this order. The picture shows this procedure.



This is not only an optional idea, it is necessary for the contraction to be canonical. For example:

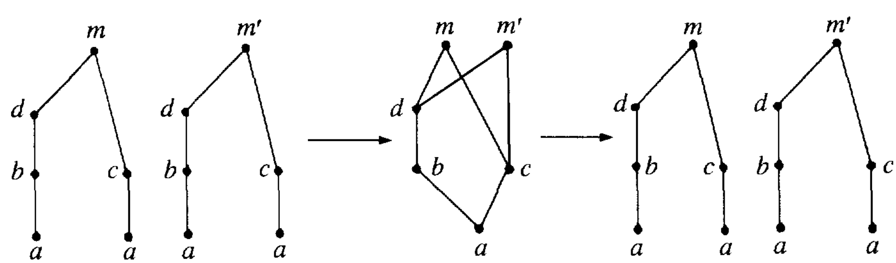


while



6.3. *First observations about the prelabeling of the level N_p, $p > 0$*

For the N_p points with the same closure it is only possible to distinguish them by means of their followers. The situation is similar to the level N_0. For instance:



(a) The points b and c are of the same level and have the same closure; however, they do not have the same follower and it is not possible to identify them because c covers b.

(b) To define an ordering in N_0 prelabels for each one of the "support branches" is fundamental in order that the desingularized of the contracting graph be the original non-labeled forest. Having fixed the prelabels b and c in the left tree, it is not possible to interchange the prelabels b and c in the right tree.

Once the N_0 points have been prelabeled, the procedure for prelabeling the N_p points is as follows: to classify these points by having an equal closure and for every one of these classes we will consider the induced forest obtained by removing each one of their equal closures by its maximal point. Then we prelabel these new N_0 points. Next, we can repeat this operation for every class and the complete procedure for $p=1,\dots,k-1$. The step $p=k=\dim(Z,H)$ would correspond to the maximal points.

6.4. *Canonical prelabeling of a forest*

Firstly, we will give the minimal number of prelabels for the level N_0, the "support branches" of the contracting graph, the ordering in the prelabeling and the identification mentioned before. Then we will extend these operations to the other levels, and finally we will give a minimal canonical contraction of a forest as sought.

The classification of a forest (Z,H) in levels $Z=\bigcup_{p=0}^{k} N_p$ with $k=\dim(Z,H)$ induces a partition in the maximal set M and it classifies the N_p -points by the levels of their followers. We will denote maximal set in level N_p by $M_p=M\cap N_p$ and the N_p-points with follower in the level N_q by $N_q^p=\{x\in N_p \mid \vec{x}\in N_q\}$ ($\vec{x}$ denotes the follower of x). Note that M_p (resp. N_q^p) can be empty for some p (resp. p and q).

Definition 6.4.1. The paths in a forest are fully described by their extremes. Thus, the path $xx_1\cdots x_nz$ can be briefly denoted by $[x,z]$.

A path $[x,y]$ with $x\in N_0$ and $y\in M$ (i.e. a maximal path) is called a *branch* of the forest. When $x\in N_0^1$ the branch is said to be a *principal branch* (i.e. $\vec{x}\in N_1$). For a branch $[x,y]$ the *budding* is defined as $b[x,y]=\{z\in N_0 \mid \vec{z}\in [x,y]\}$. The points of $b[x,y]$ are called *buds*, and z will be a *bud of level* N_q if $z\in N_0^q$ (i.e. $\vec{z}\in N_q$). The *budded branch* of $[x,y]$ is the induced subgraph in $[x,y]_b=[x,y]\cup b[x,y]$.

Proposition 6.4.2. (a) *Every branch has at least one bud.*

(b) *Two different branches with some common buds are in the same tree.*

(c) *If two different branches have a common bud, then they have in common all buds of equal or upper level.*

(d) *In every branch there is at least one bud covering all the others.*

(e) *For every branch* $[x,y]$ *there exists a principal branch* $[z,y]$ *such that* $b[x,y] \subset b[z,y]$.

(f) *The points of* $N_0 - M_0$ *are buds of principal branches.*

Proof. (a) $x \in b[x,y]$.

(b) If $[x,y],[u,v]$ are two branches and $z \in b[x,y] \cap b[u,v]$, then there exist the paths $[\vec{z},y]$ and $[\vec{z},v]$. Since there is only one path with maximal end starting in $\vec{z}$ then $y=v$. So the tree $(\bar{y},H/\bar{y})$ coincides with the tree $(\bar{v},H/\bar{v})$ and contains the branches $[x,y]$ and $[u,v]$.

(c) Since the paths are unique, $[\vec{z},y]$ and $[\vec{z},v]$ are the same path and so they have the same budding.

(d) Let q be the greatest integer for which there exists $z \in br[x,y]$ with $\vec{z} \in N_q$, then for every $v \in br[x,y]$ one has $\vec{v} \leq \vec{z}$ and so z covers v.

(e) If $[x,y]$ is principal, it is enough to take $z=x$. If $[x,y]$ is not principal, then $x \in N_0^q$ with $q > 1$, i.e., $\vec{x} \in N_q$, with $q > 1$, so there exists a path of length q starting at a point $z \in N_0^1$ and ending at $\vec{x}$. The branch $[z,y]$ is principal. On the other hand, if $u \in b[x,y]$ then $\vec{u} \in [x,y]$ and $\vec{u} \neq x$ so $\vec{u} \in [\vec{x},y]$ and x is a bud of $[z,y]$.

(f) The branch set contains $Z - M_0$; in particular, every point of $N_0 - M_0$ is a bud in some principal branch. □

Corollary 6.4.3. (a) *Every budded branch is a subgraph of a budded principal branch.*

(b) *The forest* (Z,H) *is the union of its budded principal branches and the set of disconnected points. This union is not disjoint either for the points or the arcs. Also notice that* $[x,y]_b$ *can equal* $[u,v]_b$ *even if* $[x,y]$ *is different from* $[u,v]$ *(for example if* $y=v$ *and* $\vec{x}=\vec{u}$*).*

Definition and notations 6.4.4 (*Prelabeling of the level* N_0). We will call *dimension of level* N_0 in the forest (Z,H) to the number

$$n_0 = \dim(N_0) = \max\{\mathrm{Card}(b[x,y]) \mid [x,y] \text{ is a branch of } (Z,H)\}.$$

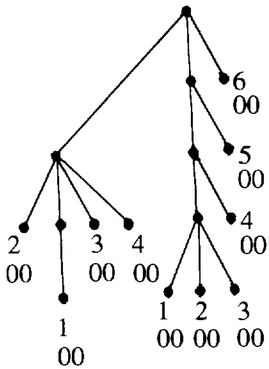
The number n_0 is the cardinality of the budding of at least a principal branch. The branches with budding of cardinal n_0 link-up in the level N_0 constituting the support of the contracted graph. In general, it will have several principal branches with budding of cardinal h with $1 \leq h \leq n_0$.

Let P be the set of principal branches and $P_h = \{[x,y] \in P \mid \mathrm{Card}\,(b[x,y]) = h\}$, for any $1 \le h \le n_0$.

(1) If $[x,y] \in P_{n_0}$, the points of $b[x,y]$ carry prelabels $x_{00}^1, x_{00}^2, \ldots, x_{00}^{n_0}$, where the subscripts indicate that they are in the level N_0. The superscripts distinguish the n_0 buds of that branch and define a sequence between the prelabels preserving the order of the levels of the followers to these n_0 buds, i.e. one has the following property

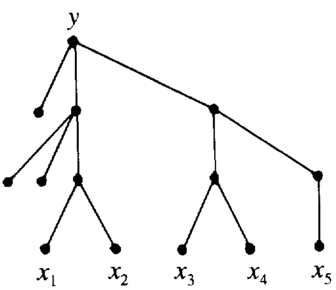
If $z, z' \in b[x,y]$, with $z = x_{00}^i \in N_0^p$ and $z' = x_{00}^j \in N_0^q$, then $p < q \Rightarrow i < j$.

It is possible that different branches of P_{n_0} have common buds. These buds have unique prelabels, exactly the last q prelabels, q being the number of common buds (Proposition 6.4.2(c)).

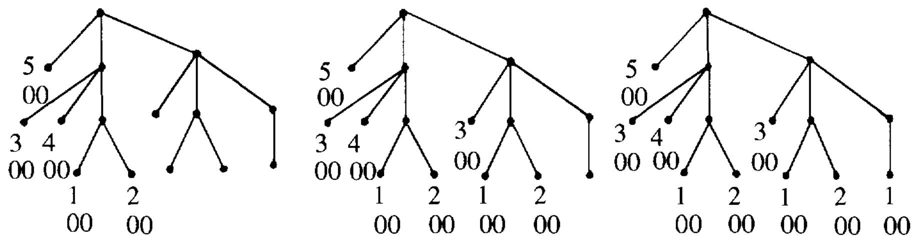


In general, the set of buds of the branches of P_{n_0} does not contain $N_0 - M_0$. If this is the case, there are buds in the forest without a prelabel. Condition (3) in Theorem 4.6 forces us to act in a descending order, i.e. recursively for $n_0 - 1 \le h \le 1$.

(2) If the buds of the branches of $P_{n_0}, P_{n_0-1}, \ldots, P_{h+1}$ have been prelabeled in that order and if $[x,y] \in P_h$, there remains a number p of buds in $[x,y]_b$ without prelabel. The prelabels of these buds will be denoted by $x_{00}^1, x_{00}^2, \ldots, x_{00}^p$. In this way the h prelabels of $[x,y]$ are $x_{00}^1, \ldots, x_{00}^p, x_{00}^{p_1}, \ldots, x_{00}^{p_{h-p}}$ where $p_1, \ldots, p_{h-p}$ is a subsequence of the sequence $p+2, p+3, \ldots, n_0$. For example, in the following graph:



$n_0 = 5$ and there are two branches $[x_1, y], [x_2, y] \in P_5$, another two $[x_3, y], [x_4, y] \in P_4$ and one $[x_5, y] \in P_3$ which will be prelabeled in an orderly way as follows:



The P_5 branches carry prelabels: $x_{00}^1, x_{00}^2, x_{00}^3, x_{00}^4, x_{00}^5$; the P_4 branches: $x_{00}^1, x_{00}^2, x_{00}^3, x_{00}^5$, and the P_3 branch: $x_{00}^1, x_{00}^3, x_{00}^5$.

(3) The M_0 points (disconnected points) will be distinguished by prelabels: $x_{00}^{-1}, \dots, x_{00}^{-m_0}$, where $m_0 = \mathrm{Card}(M_0)$.

Observation 6.4.5. The prelabeling of N_0 is unique up to automorphisms. Automorphisms act on the buds having the same follower and also on the points of M_0. If we classify the N_0 points by means of the relation "to have the same follower or to be maximal" and if the classes contain $n_1, n_2, \dots, n_r, m_0$ points, then the number of automorphisms preserving the prelabeling of N_0 is $n_1!n_2!\cdots n_r!m_0!$, where $n_1 + n_2 + \cdots + n_r \geq n_0 = \dim(N_0)$. The equality is only possible when the forest has only one principal branch.

Definition and notations 6.4.6 (*Prelabeling of the level N_p*). Firstly we must classify the N_p points by means of their closures.

For $x, y \in N_p$, x and y have *isomorphic closures* if their induced trees $(\bar{x}, Z/\bar{x})$, $(\bar{y}, Z/\bar{y})$ are isomorphic. This relationship divides N_p in classes which will be denoted by $[x_{p1}], [x_{p2}], \dots, [x_{pr_p}]$.

For every class $[x_{pi}]$, $i = 1, \dots, r_p$ we consider the induced subgraph of the forest (Z, H) in $G_{pi} = \bigcup_{x \in [x_{pi}]} x^*$, obtaining a new forest $[x_{pi}]^* = (G_{pi}, H/G_{pi})$.

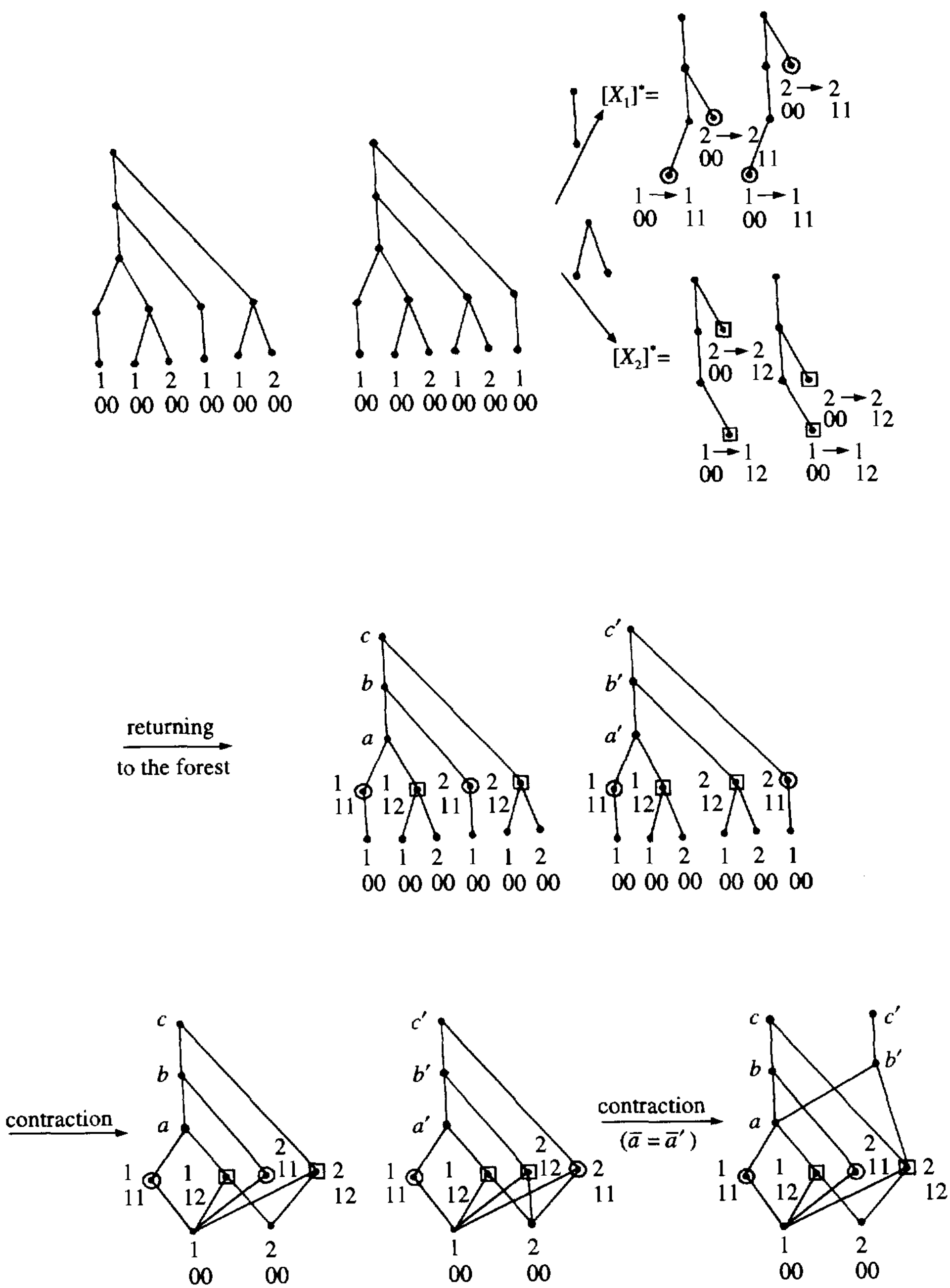
The N_0-points in $[x_{pi}]^*$ are exactly the $[x_{pi}]$-points in (Z, H) . The points of $M \cap [x_{pi}]$ are disconnected points in $[x_{pi}]^*$ and will be denoted by M_{pi}.

Let n_{pi} be the dimension of N_0 in the forest $[x_{pi}]^*$ and $m_{pi} = \mathrm{Card}(M_{pi})$. Now, we prelabel the level N_0 of $[x_{pi}]^*$ by the procedure described in definition and notations 6.4.4 and so we will give prelabels $x_{pi}^1, x_{pi}^2, \dots, x_{pi}^{n_{pi}}$ for the non-maximal points and $x_{pi}^{-1}, x_{pi}^{-2}, \dots, x_{pi}^{-m_{pi}}$ for disconnected points of $[x_{pi}]^*$ (maximals in (Z, H)). Here the notation x_{00}^j of $[x_{pi}]^*$ has been changed for x_{pi}^j as a point of the class of closure $[x_{pi}]$ in (Z, H) .

We will call *dimension of level N_p* in the forest (Z, H) the number $n_p = \sum_{i=1}^{r_p} n_{pi}$. If $m_p = \sum_{i=1}^{r_p} m_{pi} = \mathrm{Card}(M_p)$, then $n_p + m_p$ is the number of prelabels which are sufficient for the level N_p.

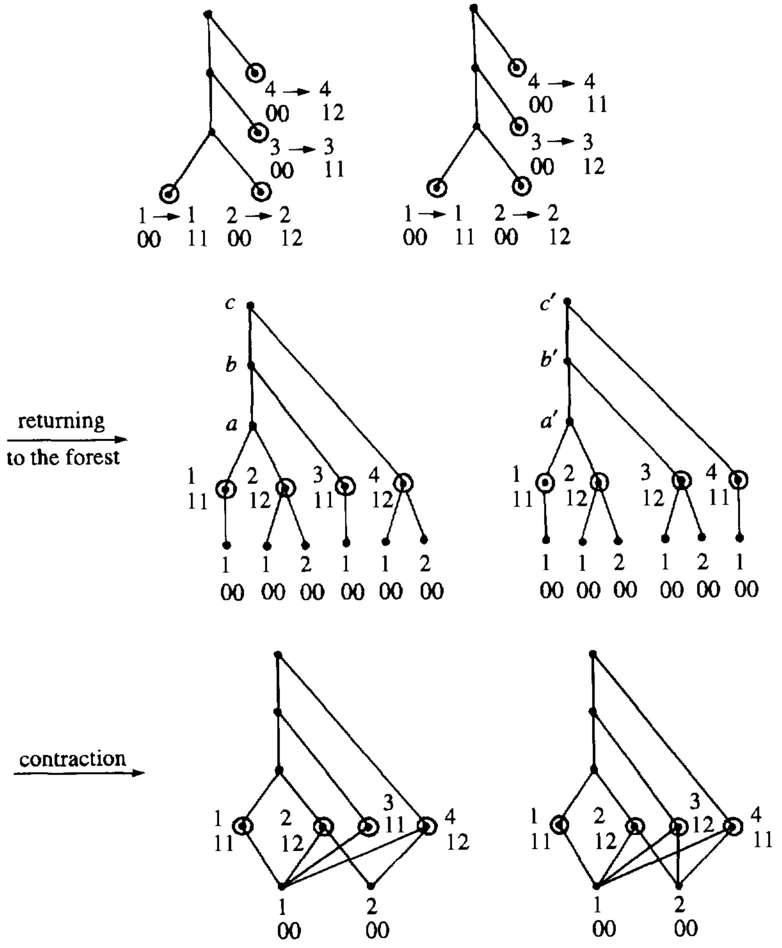
This prelabeling is unique up to automorphism. Automorphisms act in each $[x_{pi}]^*$ on the buds with the same follower and on the maximals.

Example 6.4.7. In the following pictures we show the prelabeling of the levels N_0 and N_1 of a forest (Z,H) and we perform some p-contractions:



Remark 6.4.8. Note that the separate prelabeling of the classes $[x_{pi}]$ has been essential in the above description. Also note that once the N_0 level has been labeled then, in order to label the N_1 level, prelabeling the minimal points in $(Z - N_0, H/Z - N_0)$ is not a good idea as, in this case and in general, one may not obtain minimal cardinality for the contraction and its desingularization may not even be the starting forest.

In the above example $(Z - N_0, Z/Z - N_0)$ would be:



At the moment, everything is working since these trees are isomorphic to those in the above example. However, in these trees there are 6 points in the N_1 level with pairwise different prelabels. Points in the class $[x_{12}]$ of both trees can be identified as they are isomorphic and have different followers, whereas the fact that the followers are at a distinct level, does not allow us to preserve this order in the prelabeling in the procedure.

Remark 6.4.9. We can prelabel the maximal points of (Z,H) separately, without bearing in mind either its level or its closure. If $m=m_0+m_1+\cdots+m_k$, then $m! \geq m_0!m_1!\cdots m_k!$ and if $m_p=\sum_{i=1}^{r} m_{pi}$, then $m_p! \geq \prod_{i=1}^{r} m_{pi}!$. Therefore for our prelabeling a minimal number of automorphisms are operating.

In the following paragraph we summarize the full prelabeling with its conditions.

Definition 6.4.10 (*Canonical prelabeling of a forest*). *Level* N_0 (minimals)

$$x_{00}^1, x_{00}^2, \ldots, x_{00}^{n_0}, \text{ where } n_0 = \dim(N_0) \text{ in } (Z,H) = \max\{\mathrm{Card}(br[x,y]) : [x,y] \text{ is a branch of } (Z,H)\}.$$

$$x_{00}^{-1}, x_{00}^{-2} \ldots, x_{00}^{-m_0}, \text{ where } m_0 = \mathrm{Card}(M_0) \text{ (disconnected points of } (Z,H)).$$

Level N_p

For every $p=1,\ldots,k-1$ we have the closure classes $[x_{p1}],\ldots,[x_{pr_p}]$ and for every class $[x_{pi}]$ we have

$$x_{pi}^1, x_{pi}^2, \ldots, x_{pi}^{n_{pi}}, \text{ where } n_{pi} = \dim(N_0) \text{ in } [x_{pi}]^*$$
$$\text{with } [x_{pi}]^* = (G_{pi}, H/G_{pi}) \text{ and } G_{pi} = \bigcup_{x\in[x_{pi}]} x^*.$$

$$x_{pi}^{-1}, x_{pi}^{-2}, \ldots, x_{pi}^{-m_{pi}}, \text{ where } m_{pi} = \mathrm{Card}(M_{pi}) = \mathrm{Card}(M \cap [x_{pi}]).$$

Level N_k (maximals)

We have the closure classes $[x_{k1}],\ldots,[x_{kr_k}]$ and for every class $[x_{ki}]$:

$$x_{ki}^{-1}, x_{ki}^{-2}, \ldots, x_{pi}^{-m_{ki}}, \text{ where } m_{ki} = \mathrm{Card}(M_{ki}) = \mathrm{Card}(M \cap [x_{ki}]).$$

The necessary number of prelabels is $n+m$, where

$$n = \sum_{p=0}^{k-1} \dim(N_p) = n_0 + \sum_{p=1}^{k-1} \left(\sum_{i=1}^{r_p} n_{pi}\right),$$

$$m = \mathrm{Card}(M) = \sum_{p=0}^{k} m_p = m_0 + \sum_{p=1}^{k} \left(\sum_{i=1}^{r_p} m_{pi}\right).$$

The set of these $n+m$ prelabels will be denoted by E.

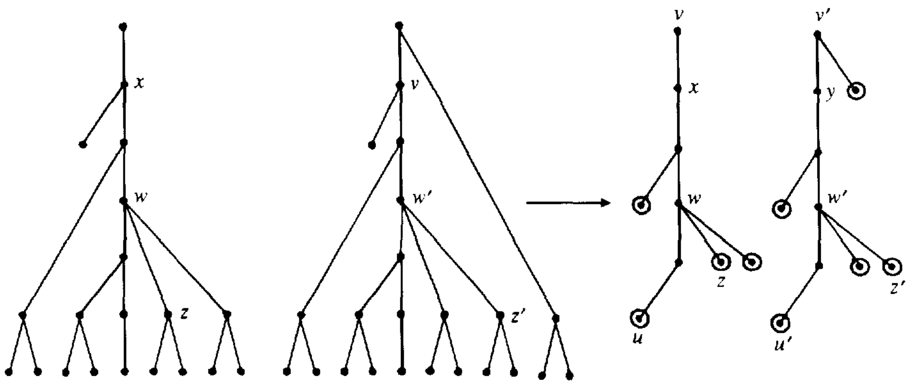
Let $c: Z \to E$ be the surjective map carrying each point $x \in Z$ into its prelabel $x_{pi}^j \in E$. This prelabeling in the forest (Z,H) is canonical by construction.

Theorem 6.4.11. *The map* $c: Z \to E$ *is c-contractive.*

Proof. By construction if $x,y \in Z$ are points with the same prelabel, then they are not maximals and nor does one cover the other; moreover, they have isomorphic closures. We will construct an isomorphism $\Delta: (\bar{x}, H/\bar{x}) \to (\bar{y}, H/\bar{y})$ such that $c(z)=c(\Delta(z))$, for any $z \in \bar{x}$. (Theorem 4.6).

(a) If $x, y \in N_1$ then $\bar{x} - \{x\} \subset N_0^1$ and $\bar{y} - \{y\} \subset N_0^1$. Since these points are buds with followers in N_1 they cannot be in common with other principal branches, and so they receive the first p prelabels of the n_0 necessary in the level N_0, where $p = \mathrm{Card}(\bar{x} - \{x\}) = \mathrm{Card}(\bar{y} - \{y\})$. These prelabels are $x_{00}^1, x_{00}^2, \ldots, x_{00}^p$ both for the points of $\bar{x} - \{x\}$ and the points of $\bar{y} - \{y\}$. Any bijection between $\bar{x} - \{x\}$ and $\bar{y} - \{y\}$ is an isomorphism since all points are disconnected; therefore, we can take the map $\Delta : \bar{x} - \{x\} \to \bar{y} - \{y\}$ which identifies the prelabels of both sets.

(b) If $x, y \in N_q$, $q > 1$ and $z \in \bar{x}$, let z' be the image of z by any isomorphism $\Phi : \bar{x} \to \bar{y}$. Let $c(z) = x_{pi}^j$ and $c(z') = x_{p'i'}^{j'}$ be the prelabels of z and z'. The restriction of Φ to $\bar{z}$ is an isomorphism between the induced subgraphs in $\bar{z}$ and $\bar{z}'$ and therefore $i = i'$ and $p = p'$. We will denote the corresponding closure class by $[x_{pi}]$. We will now prove that $j = j'$.



(Z, H) $[Xp_i]^*$

The prelabels of z and z' as N_0 points of $[x_{pi}]^*$ are $x_{00}^j, x_{00}^{j'}$ respectively.

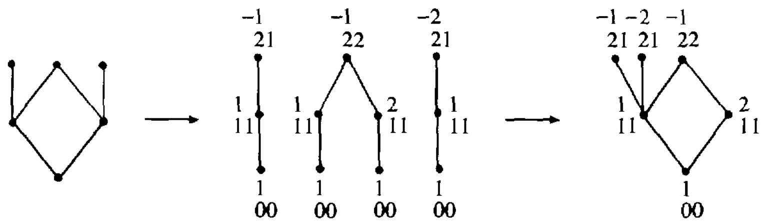
As $\bar{x}$ and $\bar{y}$ are disjoint trees, then $\bar{x} \cap [x_{pi}]^*$ and $\bar{y} \cap [x_{pi}]^*$ are disjoint. If z and z' are respective buds for the principal branches $[u, v] \in P_h$ and $[u', v'] \in P_h$, with (possibly) $h \neq h'$, then $\bar{x} \cap [x_{pi}]^*$ and $\bar{y} \cap [x_{pi}]^*$ are isomorphic lower parts of the branches $[u, v]$ and $[u', v']$, therefore if $\vec{z} \in N_q$ also $\vec{z}' \in N_q$ and there are the same number of buds with followers in lower levels to N_q in the two branches; so z and z' occupy the same position between the buds of $[x_{pi}]^*$ (module an automorphism acting on the buds of $[x_{pi}]^*$ with follower in N_q) and hence $j = j'$.

(c) Finally, we need to modify the isomorphism Φ by means of a permutation identifying the prelabels of the above mentioned buds of $[x_{pi}]^*$ with their images. This is necessary for every pair of homologous points $z \in \bar{x}$ and $z' = \Phi(z) \in \bar{y}$ and we must do it from the bottom up in order. Properly speaking, if $w \in N_j, j < q$, for every closure class $[x_{pi}]$, with $p < j$, we consider the sets $A_{pi} = \{t \in [x_{pi}] \mid \vec{t} = w\}$ and $A'_{pi} = \{t' \in [x_{pi}] \mid \vec{t}' = w' = \Phi(w)\}$. A_{pi} and A'_{pi} receive the same prelabel set $x_{pi}^{k+1}, \ldots, x_{pi}^{k+a}$ where $a = \mathrm{Card}(A_{pi}) = \mathrm{Card}(A'_{pi})$, and there exists a permutation $p : A_{pi} \to A_{pi}$ such that

$(\Phi/A_{pi}) \circ p$ identifies the prelabels of A_{pi} points with their images in A'_{pi}. We perform this operation for every $p < j$ and each i such that $[x_{pi}]$ is a closure class with follower in z. The operation is extended to all points w of the same level N_j and we must go over the levels in order from $j = 1$ to $j = q - 1$. From these modifications we get the isomorphism Δ that we were looking for. □

Definition 6.4.12 (*Minimal canonical contraction of a forest*). If (Z,H) is a forest canonically prelabeled by means of the contractive map $c: Z \to E$ of Definition 6.4.10 and $K = \{(i,j) \in E_x E \mid (x,y) \in H \text{ and } c(x) = i,\ c(y) = j\}$, then the graph (E,K) is the c-contraction of (Z,H) and it will be called the *canonical contraction* of the forest (Z,H). The graph (E,K) is clearly prelabeled by E.

The construction of the canonical prelabeling guarantees that the contracted graph (E,K) has a minimal number of points and arcs between the class of reduced graphs desingularizing in (Z,H), then in particular the graph (E,K) is a minimal element in the partial ordering of Definition 5.6. However, it is not the only graph with these properties. The picture shows a reduced graph, its desingularized forest with the canonical prelabeling and its prelabeled canonical contraction. Two different contractions of the central forest are exhibited which have the same number of points and arcs.



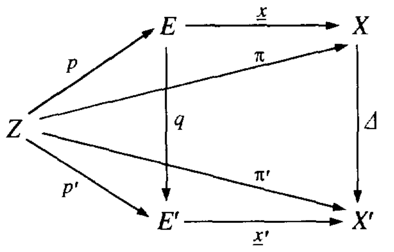
7. More about the structure of the isodesingularized reduced graph set

The reduced graph associated to the partial ordering in Definition 5.6 will be denoted by $\mathrm{Iso}(Z,H)$. The following results are also consequences of the result on contractive mappings (Theorem 4.6).

Theorem 7.1. *If $\Delta : (X,G) \to (X',G')$ is a birational reduced graph morphism with* $\mathrm{Card}(X) = m$ *and* $\mathrm{Card}(X') = n$, *then for every q, with $n < q < m$, there exists a reduced graph (X'',G''), with* $\mathrm{Card}(X'') = q$, *and there exist birational morphisms $\Delta_1 : (X,G) \to (X'',G'')$ and $\Delta_2 : (X'',G'') \to (X',G')$ such that $\Delta_2 \circ \Delta_1 = \Delta$.*

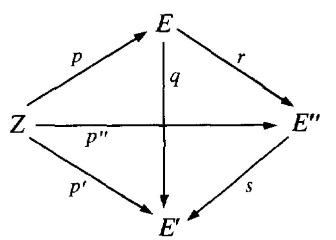
Proof. It will be enough to prove it for the case $q = n + 1$.

Let $\underline{x}: E \rightarrow X$ and $\underline{x}': E' \rightarrow X'$ be respective labelings for the reduced graphs (X,G) and (X',G') being desingularized in the forest (Z,H). These labelings induce prelabelings $p: Z \rightarrow E$ and $p': Z \rightarrow E'$ over the forest (Z,H). The map Δ induces surjective map $q: E \rightarrow E'$ such that $q \circ p = p'$. So we have the diagram:

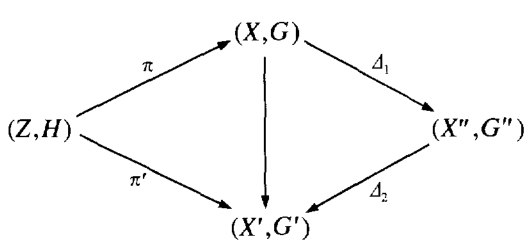


where $\pi = \underline{x} \circ p$ and $\pi = \underline{x}' \circ p'$ are the desingularization morphisms induced by the prelabelings p and p'.

In particular, one has that two points in Z with the same E-prelabel also have the same E'-prelabel. Take two points in Z at the greatest possible level with the same E'-prelabel and different E-prelabel (the existence follows from the fact $n < m$). Let E'' be the quotient set obtained by identifying the E-prelabels plus the identification of $p(x)$ to $p(y)$. The mapping $q: E \rightarrow E'$ factorizes through E'' , i.e. one has a diagram:



By construction it follows that $p'' = r \circ p$ is a contractive prelabeling (Theorem 4.6), therefore one has a commutative diagram of birational morphisms:



□

From the structure of the reduced graph $\mathrm{Iso}(Z,H)$ we know that the forest (Z,H) is the maximum of this graph and that there are in general several minimals with a minimal number of points (picture 6.4.12). From the above theorem one can also deduce the following consequences:

Corollary 7.2. *Assume that* (Z,H) *is a forest with* $\mathrm{Card}(Z)=m$.

(1) *For every* $(X,G) \in \mathrm{Iso}(Z,H)$, *with* $\mathrm{Card}(X)=n$, *there is at least one path with maximal end in* $\mathrm{Iso}(Z,H)$ *starting at* (X,G) *of length* $m-n$.

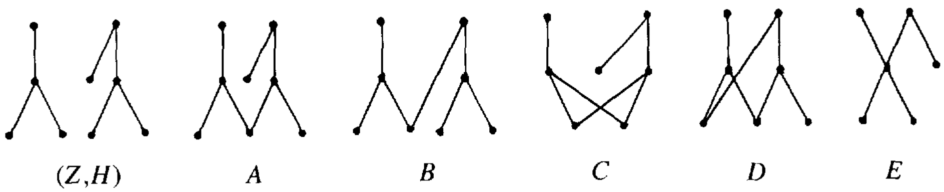
Every path with maximal end starting at (X,G) *has a length equal to* $m-n$.

(2) *If* (E,K) *is the minimal canonical contraction of* (Z,H) *and* $s=\mathrm{Card}(E)$ *then* $m-s$ *is the dimension of the graph* $\mathrm{Iso}(Z,H)$.

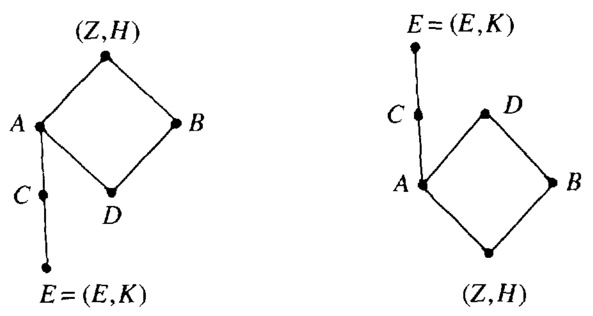
(3) *The arcs in the dual graph of* $\mathrm{Iso}(Z,H)$ *are joining points in consecutive levels of* $\mathrm{Iso}(Z,H)$.

Graphs (X,G) *at the level* N_p *in the dual graph of* $\mathrm{Iso}(Z,H)$ *are exactly those graphs with* $\mathrm{Card}(X)=m-p$.

Example 7.3. The following example shows us a forest (Z,H) and all its contractions: two with 8 points (A and B), another two with 7 (C, D) and one with 6 points (E). D is a minimal element in $\mathrm{Iso}(Z,H)$ and E is the minimal canonical contraction.



The last picture shows the reduced graph $\mathrm{Iso}(Z,H)$ and its dual.



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